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Fins with temperature dependent surface heat flux—I. Single heat transfer mode

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Abstract—Involved in all possible types of heat transfer, the thermal characteristics of a single fin with and without heat transfer at its tip is studied. The heat transfer coefficient is assumed to vary with a power-law-type formula. Based on the value of the exponent, the problem is categorized into three regimes. The temperature distribution and heat transfer rate are given in implicit forms and solved exactly. No solution is found when the fin parameter exceeds a specific value for a negative exponent. Besides, two solutions are detected when the fin parameter is smaller than this value, which is related to the thermal instability. Results from analysis are compared with experimental data.

INTRODUCTION

EXTENDED surfaces are frequently used in engineering to increase the rate of heat transfer between an object and its surrounding fluid. Typical applications include cooling of nuclear fuel rods, electrical equipment, and evaporators. For years, fins have been used extensively in situations when the heat transfer coefficient is nearly constant. In those cases, the temperature distribution as well as the heat transfer rate can be obtained analytically [1].

Recently, the study of fins in boiling liquids has been increasing enormously. In this situation, the heat transfer coefficient is no longer uniform. It varies with the temperature difference between the heater surface and the adjacent boiling liquid in a nonlinear manner. However, the value of this heat transfer coefficient may be governed by an approximate power-law-type temperature dependence. Under these circumstances, the equation for temperature becomes strictly nonlinear even in the simplest one-dimensional analysis.

A fin with an insulated end has been studied by many investigators. Lai and Hsu [2] presented a simple model to compute the base heat flux and the length of the nucleate boiling section. Their results were compared favorably with Haley and Westwater's data [3]. Later on, Mehta and Aris [4, 5] solved the same equation applied to porous catalysts with chemical reaction. All orders of reaction were taken into account and the solutions were given in terms of a hypergeometric function. Recently, Unal [6] derived some closed-form solutions for the temperature distribution in fins. Sen and Trinh [7] have made use of Mehta and Aris's model to compute the heat transfer rate of a single fin cooled by natural convection, nucleate boiling, and radiation. However, the cases of film and transition boiling were ignored.

The heat flux in transition boiling decreases with the increase of wall superheat. It is very difficult to establish a stable mode in boiling experiments. Bui and Dhir [8] observed the transition boiling of water on a vertical copper surface and concluded that steady state data in transition boiling near the minimum heat flux region is attainable. According to their data, however, only transient data were obtained when a heat flux is higher than twice that of the minimum heat flux. Similar results were reported by Liaw and Dhir [9] in quenching experiments. The mystery of transition boiling being unstable is intriguing to study. Therefore the cases with negative exponents are especially noted in this study.

An insulated boundary condition is usually prescribed at the end of a fin due to its low temperature and negligible heat loss. It is worthwhile to perform such an analysis in a long fin or a rod between two hot walls. However, in boiling heat transfer, the low-temperature fin tip may be subject to a high heat transfer coefficient. Therefore, both cases of a fin with and without insulation at its end are included in this study.

The objective of this study is to investigate a fin governed by a power-law-type heat transfer coefficient. Based on the value of the exponent, a one-dimensional model with a prescribed heat flux at the end of the fin is used to calculate the temperature distribution and the rate of heat transfer from the base. In addition, linear stability analysis is also performed to facilitate a better understanding of the results. The present work is to serve as a framework with which the coexistence of many boiling modes on a long fin can be assessed.

MATHEMATICAL ANALYSIS

A straight fin with a uniform cross section is now considered. The surface heat flux along the fin is con-

NOMENCLATURE

A	cross-section area of fin
a	dimensional constant, equation (1)
C_n	constants
F	hypergeometric function
G	Dawson's integral, equation (12)
h	heat transfer coefficient
k	thermal conductivity of fin
L	length of fin
m	power-law exponent, equation (1)
N	fin parameter, equation (3)
N_{max}	maximum fin parameter existing for $m \leqslant 0$
N_0	defined in equation (28)
P	periphery of fin
Q	dimensionless heat flux
q	heat flux
T	temperature
t	time
X	dimensionless distance from fin tip.

Greek symbols

α	aspect ratio, $A/(P \cdot L)$
β	constant, equation (7)
θ	dimensionless temperature
θ'	infinitesimal non-steady disturbance
κ	thermal diffusivity
λ	eigenvalue
τ	dimensionless time.

Subscripts

b	fin base
0	fin end
∞	ambient liquid.

sidered to exhibit a power-law-type dependence on the temperature difference between the fin and the ambient fluid, i.e.

$$q = a(T - T_\infty)^m. \tag{1}$$

For simplicity, a one-dimensional steady state heat conduction equation is analyzed. Introducing the dimensionless variables $\theta = (T - T_\infty)/(T_b - T_\infty)$ and $X = x/L$, it is written as

$$\frac{d^2\theta}{dX^2} - N^2\,\theta^m = 0 \tag{2}$$

where the fin parameter, N, is defined as

$$N = \sqrt{\left[\frac{aPL^2}{kA}\cdot(T_b - T_\infty)^{m-1}\right]} = \sqrt{\left(h_b\cdot\frac{PL^2}{kA}\right)}. \tag{3}$$

In the above equation, P, L, k, and A represent, respectively, the periphery, length, conductivity, and cross-section area of the fin. The value of m was reported to vary in a wide range between -3 and 5.5 [10, 11] depending on the mode of boiling. For example, the exponent m may take the respective values -3, 0.75, 1, 3, and 4, depending on whether the fin subject to transition boiling, laminar film boiling or condensation, convection, nucleate boiling, and radiation into free space at zero absolute temperature.

An appropriate set of boundary conditions is

$$\theta(1) = 1 \tag{4}$$

$$\frac{d\theta(0)}{dX} = Q_0 \tag{5}$$

where Q_0 is the dimensionless heat transfer rate prescribed at the fin tip. Apparently, when Q_0 equals zero, the problem reduces to the special case of a fin with an insulated tip.

$m > -1$. Integration of equation (2) yields

$$\frac{d\theta}{dX} = N\cdot\sqrt{\left[\frac{2}{m+1}\cdot(\theta^{m+1} - \beta)\right]} \tag{6}$$

where β is denoted as

$$\beta = \theta_0^{m+1} - \frac{m+1}{2}\cdot N^{-2}\cdot Q_0^2. \tag{7}$$

Integrating equation (6) again, we get

$$N\cdot X = \sqrt{\left(\frac{2}{m+1}\cdot(\theta^{-m+1} - \theta^{-2m}\beta)\right)} \cdot F\left[1, \frac{m}{m+1}; \frac{3}{2}; 1 - \beta\cdot\theta^{-m-1}\right] - C_1 \tag{8}$$

and

$$C_1 = Q_0\cdot N^{-1}\cdot\theta_0^{-m}\cdot F\left[1, \frac{m}{m+1}; \frac{3}{2}; 1 - \beta\cdot\theta_0^{-m-1}\right] \tag{9}$$

where F represents the hypergeometric function [12]. The hypergeometric function converges on condition that the absolute value of the last term in the brackets is less than unity. The tip temperature is obtained implicitly by applying equation (4)

$$N = \sqrt{\left(\frac{2}{m+1}\cdot(1-\beta)\right)}\cdot F\left[1, \frac{m}{m+1}; \frac{3}{2}; 1-\beta\right] - C_1. \tag{10}$$

It is clear that β tends to θ_0^{m+1}, and C_1 vanishes as Q_0 approaches zero.

$m = -1$. If m exactly equals -1, equation (2) can be solved by some algebraic manipulation. A subsequent solution is derived as

$$X = \frac{\sqrt{2}}{N} \cdot [\theta \cdot G(z) - \theta_0 \cdot G(z_0)] \tag{11}$$

where $G(z)$ is called Dawson's integral [12] and is defined as

$$G(z) = e^{-z^2} \cdot \int_0^z e^{u^2}\, du \tag{12}$$

with

$$z = \sqrt{(\ln(\theta/\theta_0) + z_0^2)} \tag{13}$$

and

$$z_0 = Q_0/(\sqrt{(2)}N). \tag{14}$$

Once the fin parameter, N, is known, the tip temperature can be calculated by imposing equation (4) on equations (11) and (13). Hence, the temperature distribution of the fin can be obtained by solving equations (11) and (13) simultaneously.

$m < -1$. Similar to the procedures as those stated earlier in the section for $m > -1$, the resulting equation for the temperature distribution is obtained as

$$N \cdot X = \theta \cdot \sqrt{\left[\frac{2}{m+1} \cdot (\theta^{m+1}\beta^{-2} - \beta^{-1})\right]} \cdot F\left[1, \frac{m+3}{2(m+1)}; \frac{3}{2}; 1 - \beta^{-1}\theta^{m+1}\right] - C_2 \tag{15}$$

where

$$C_2 = Q_0 \cdot N^{-1} \cdot \theta_0 \cdot \beta^{-1} \cdot F\left[1, \frac{m+3}{2(m+1)}; \frac{3}{2}; 1 - \beta^{-1}\theta_0^{m+1}\right] \tag{16}$$

with the tip temperature, θ_0, satisfying

$$N = \sqrt{\left(\frac{2}{m+1} \cdot (\beta^{-2} - \beta^{-1})\right)} \cdot F\left[1, \frac{m+3}{2(m+1)}; \frac{3}{2}; 1 - \beta^{-1}\right] - C_2. \tag{17}$$

In the above equations, β is exactly the same as that defined in equation (7). In a similar manner, β tends to θ_0^{m+1} and C_2 approaches zero for a fin with an insulated tip. In a special case of $m = -3$, the hypergeometric function F equals 1, and equation (15) is thus reduced to

$$\theta = \sqrt{\left[(Q_0 \cdot X + \theta_0)^2 + N^2 \cdot X^2 \cdot \theta_0^{-2}\right]} \quad \text{for } m = -3. \tag{18}$$

Dimensionless heat transfer rate

The quantity of the energy transferred from the base of a fin is of great interest. The dimensionless temperature gradient at the fin base, $d\theta(1)/dX$, is denoted as Q_b. It can be easily established by using the results of equations (4) and (6).

$$Q_b = \sqrt{\left(\frac{2}{m+1} \cdot N^2 \cdot (1 - \theta_0^{m+1}) + Q_0^2\right)} \quad \text{for } m \neq -1 \tag{19}$$

and

$$Q_b = \sqrt{(-2N^2 \cdot \ln\theta_0 + Q_0^2)} \quad \text{for } m = -1. \tag{20}$$

A fin with a free end

In the case when the tip of a fin is exposed to heat transfer of the same mode as that at the lateral surface, the heat transfer rate is written as

$$Q_0 = \alpha N^2 \cdot \theta_0^m \tag{21}$$

where α stands for the aspect ratio and is equal to $A/(P \cdot L)$. It should be pointed out that $\alpha = 0$ signifies a finite fin with insulated tip or an infinitely long fin.

Linear instability

An infinitesimal and arbitrary disturbance, θ', is usually superimposed on the temperature profile [13]. The linearized disturbance form of equation (2) is

$$\frac{\partial\theta'}{\partial\tau} = \frac{\partial^2\theta'}{\partial X^2} - N^2 \cdot m\theta^{m-1}\theta' \tag{22}$$

with $\tau = \kappa t/L^2$. The above equation can be solved by the method of separation of variables. For a fin with a free end, a solution of the form, $\theta'(X, \tau) = \phi(X) \cdot \psi(\tau)$, is assumed and the final solution becomes

$$\psi = e^{\lambda_n \tau} \tag{23}$$

with

$$\lambda_n = -\mu_n^2 - mN^2\theta^{m-1} \tag{24}$$

where λ_n are the characteristic values and μ_n are the roots of the following equation

$$\mu_n + m\alpha N^2 \cdot \theta_0^{m-1} \cdot \tan(\mu_n) = 0. \tag{25}$$

It is obvious that the system will become unstable for any characteristic value with a positive real part. Hence the criterion of stability is obtained as

$$-mN^2\theta^{m-1} \leqslant \mu_n^2 \tag{26}$$

where the θ is the dimensionless temperature of any part in the fin. Since the temperature is non-negative, it is clear that equation (26) always hold as long as $m \geqslant 0$. However, for $m < 0$, the maximum value of the left-hand side of equation (26) occurs at the fin tip. Hence equation (26) reduces to

$$-mN^2\theta_0^{m-1} \leqslant \mu_n^2 \quad \text{for } m < 0. \tag{27}$$

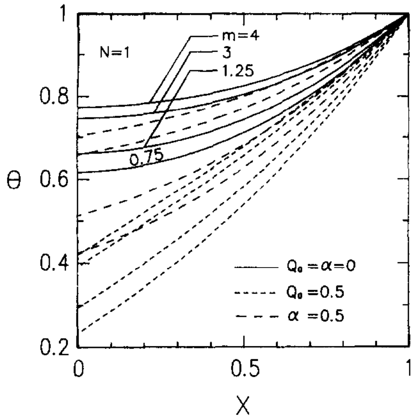

FIG. 1. Temperature profiles.

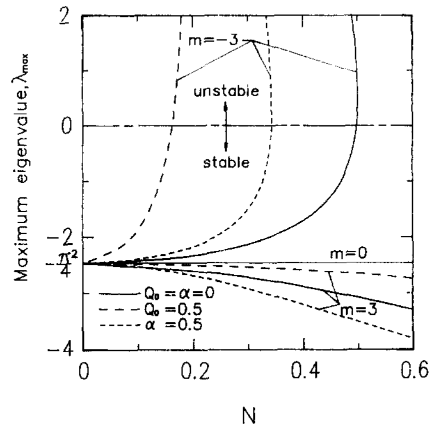

FIG. 3. Dependence of maximum eigenvalue on fin parameter.

RESULTS AND DISCUSSION

As stated earlier, Q_0 is the rate of heat transfer applied to the tip of a fin, whereas α is an index of the geometry. Both of them are studied parametrically. The temperature profiles for some selected values of α and Q_0 are displayed in Fig. 1. It is seen that the temperature decreases along the extended distance and the temperature drop is larger in the presence of heat transfer at the end. The temperature profiles for $m = 4$, 3, 1.25, and 0.75 appear from top to bottom. In accordance with equation (21), an increment in α leads to the increase of Q_0, and $\alpha = 0$ coincides with $Q_0 = 0$. The dependence of N on θ_0 is displayed in Fig. 2 for $m = 3$. A long and slender fin corresponds to a larger N which would result in a low tip temperature. Because N is proportional to L, θ_0 will decrease to zero as N approaches infinity for an infinitely long fin. In this situation, no energy is transferred at the end. With heat transfer applied at the end, the largest N is limited. In other words, if a fin is very long, it may fail to carry the prescribed Q_0. In a limiting case, the thermal conductance through the fin is perfect at $Q_0 = 1$.

As for the film boiling and film condensation heat transfer, the exponent, m, usually falls between 0 and

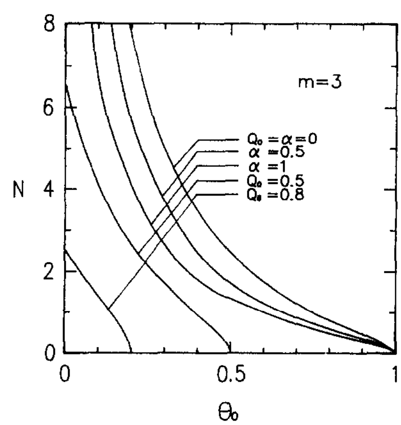

FIG. 2. Influence of Q_0 and α on fin parameter for $m = 3$.

1. As before, θ_0 decreases with an increase in L or N. As a consequence, the θ_0 drops to the ambient temperature at a finite N_0 which is calculated as

$$N_0 = \frac{\sqrt{(2(1+m))}}{1-m} \quad \text{for } 1 > m > 0. \tag{28}$$

This is the maximum available N even though no heat transfer is applied to the end. For $N \geqslant N_0$, the temperature will drop to zero somewhere on the fin, and the free end shows no effect on it. However, when $0 \geqslant m > -1$, a negative temperature might appear during the mathematical calculation due to the large heat sink at the end portion. In fact, the heat transfer process will not continue up to this situation. For example, film boiling ceases at the moment of film collapse, and then it changes to another heat transfer mechanism.

The maximum eigenvalue given by equation (24) is plotted in Fig. 3. When $Q_0 = 0$ or $\alpha = 0$, the smallest μ_n is found to be $\pi/2$. It is always stable for $m \geqslant 0$. However, there exists an upper limit of N when m is negative. As discussed in Fig. 1, the tip temperature is lowered with increasing Q_0. It leads to a large value in the last term of equation (24) and results in a larger eigenvalue. This reveals that the stability condition is under constraint when heat transfer is applied to the end of the fin. A similar effect caused by α is detected, but the influence is not so pronounced.

The N–θ_0 curves for $m = -3$ are displayed in Fig. 4. All the profiles in this figure are convex upward. There exists a peak point, referred as N_{max}, that no solution is found when N is greater than N_{max}. It is interesting to observe that two distinct solutions of θ_0 are obtained from equation (17) at a given fin parameter smaller than N_{max}. Through the stability criterion as given in equation (27), the solution with a smaller θ_0 is subject to an unstable state, whereas a larger θ_0 is stable and possible to occur spontaneously. The left part of the profiles in the unstable modes are therefore plotted with dash lines as shown in Fig. 4. The line of neutral stability is found to be the same as the locus of N_{max}. Besides, N_{max} decreases as α or Q_0

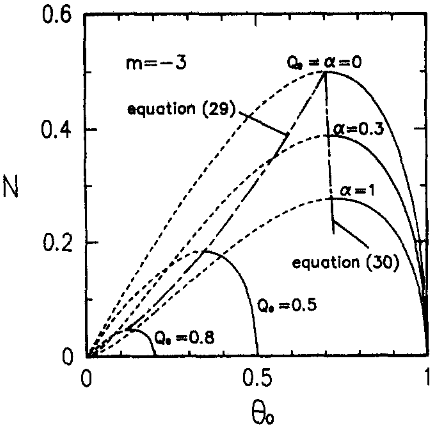

FIG. 4. Influence of Q_0 and α on fin parameter for $m = -3$.

increases, and it can be easily calculated from the corresponding θ_0 which is termed as $(\theta_0)_{max}$. From equations (4) and (18), it is expressed as

$$(\theta_0)_{max} = \frac{1}{4}\cdot\left[\sqrt{(Q_0^2+8)}-3Q_0\right] \quad (29)$$

or

$$(\theta_0)_{max} = \left[\frac{1+4\alpha-8\alpha^2+\sqrt{(1+8\alpha+36\alpha^2+80\alpha^3+64\alpha^4)}}{4+16\alpha}\right]^{1/2} \quad (30)$$

and is also shown in Fig. 4 for $m = -3$.

Regarding the value of N, no constraint is imposed on the cases of $m \geqslant 1$. However, a critical value, N_0 or N_{max} appeared in the above analysis for $1 > m$. Figure 5 shows the variation of N_{max} and its corresponding tip temperature for Q_0 and α at 0 and 0.5. In this figure, N_{max} increases with m representing that a stricter condition should be satisfied at a smaller m. This can be easily seen from equation (27). When m is negative, a fin cannot operate steadily with a fin parameter larger than N_{max}. For example, if a fin is in transition boiling, the steady state can exist only on a very short fin with a small N. This reveals the fact that steady state might be achieved at low heat transfer coefficient regime which is close to the film boiling curve, but is no longer attainable at the upper regime neighboring nucleate boiling. It is consistent with what has been observed in the literature [8, 9], such as some steady state data in the early stage of transition boiling during quenching experiments. For a fin with an insulated tip, an approximate relation is recommended as

$$(\theta_0)_{max} = 2^{1/(0.59m-0.23)} \qquad \text{for } Q_0 = 0 \quad (31)$$

which is accurate up to 0.01% for $-6 \leqslant m \leqslant -1$ in comparison with the exact solutions plotted in Fig. 5.

The quantity of most physical interest is the rate of heat transfer from the base of the fin. The results of equations (19) and (20) are illustrated in Fig. 6 at $m = 3, 0.75$, and -3. In general, Q_b increases with N, and the curve is steeper at a smaller m. In the case of a fin at a specified operating condition (constant N), Q_b is smaller at a larger m for its lower local heat transfer rate at the end portion. In addition, according to equation (19), with heat transfer at the end, the total heat transfer rate is increased due to the superimposition of Q_0 on it. While m is negative, two Q_bs are obtained at a fin parameter. However, the upper half part of the curve is unattainable and is plotted in dash lines.

EXPERIMENTS

Cylindrical copper fins with a diameter of 2.5 cm and various lengths served as test pieces. The test liquids are chosen to be distilled water and isopropyl alcohol. All of the data presented in this paper were obtained when the whole fin was governed by only a single mode of boiling heat transfer, i.e. nucleate boiling or film boiling. The test apparatus and the experimental procedure employed were the same as those described in the companion paper [14].

Validation of the model with experimental data

A cylindrical fin with a length of 8.5 cm is tested in boiling water. The measured temperatures at the

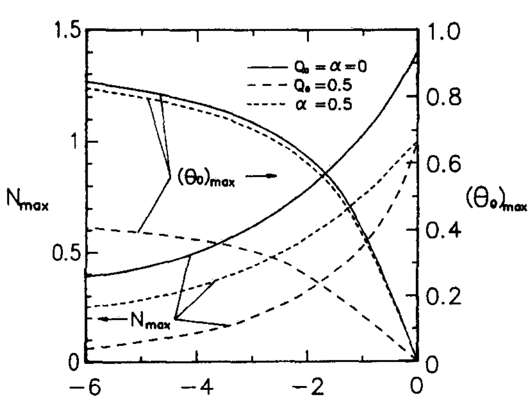

FIG. 5. N_{max} and its corresponding tip temperature.

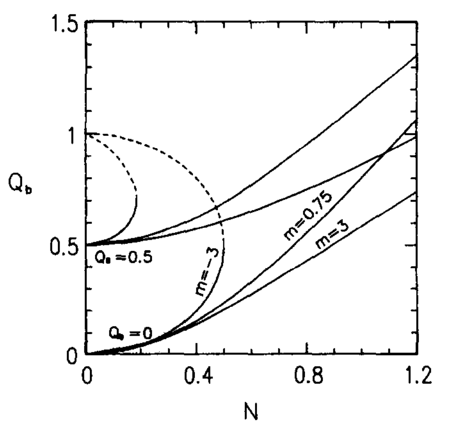

FIG. 6. Dependence of Q_b on fin parameter.

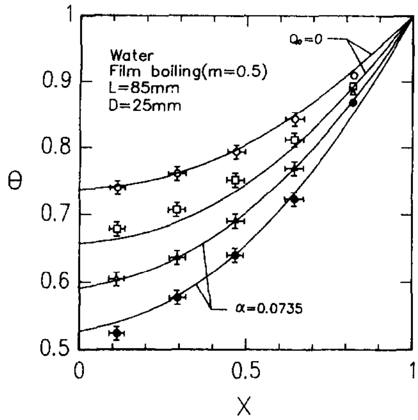

FIG. 7. Comparison of measured and predicted temperatures in film boiling with water. (○ : $N = 0.77$; □ : $N = 0.90$; Δ : $N = 0.94$; ● : $N = 1.03$.)

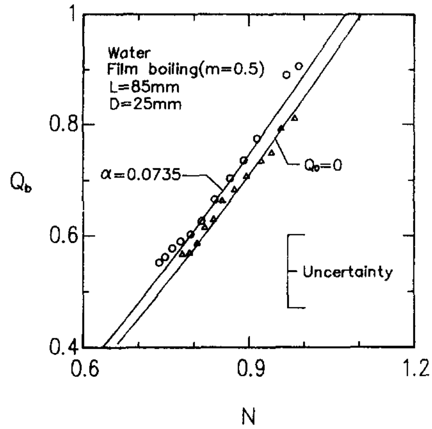

FIG. 9. Comparison of predicted and measured heat transfer rates from fin base. (○ : uninsulated end; Δ : insulated end.)

centerline of the fin are plotted in Fig. 7 for various locations. While plotting the data, the range of uncertainty in the data is also marked. In obtaining the prediction from the present model, the fin parameter based on the measured temperature at the fin base was used. The upper two curves are obtained when ΔT_bs are at 254 and 140 K with an insulated end, whereas the lower profiles are with the end exposed to the boiling liquid at $\Delta T_b = 90$ and 84 K. The data are observed to correlate well with the prediction by setting $m = 0.5$ in all the cases.

Figure 8 shows the predicted temperature distribution obtained with isopropyl alcohol for end-insulated fins at various lengths. The values of m are chosen to be 1 and 3 in film and nucleate boiling respectively such as that performed by Unal [15]. As plotted in solid lines for film boiling, the maximum difference between the model and experimental data does not exceed 8%. Figure 8 also shows the corresponding experimental and theoretical temperature distributions with nucleate boiling over the entire fin. Good agreement between them is observed.

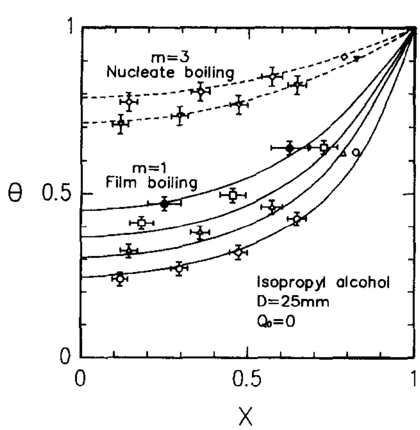

FIG. 8. Comparison of measured and predicted temperature profiles of end-insulated fins in nucleate boiling and film boiling with isopropyl alcohol. (∇, ○ : $L = 85$ mm; ◇, Δ : $L = 70$ mm; □ : $L = 55$ mm; ● : $L = 40$ mm.)

Figure 9 shows the normalized total heat transfer rate as a function of N, which varies with the operating condition prescribed at the base of the fin. In the experiments of the 8.5 cm long fin with water in film boiling, N falls between 0.7 and 1. The present analysis results in approximately linear dependence of Q_b in the entire range of N. The model predictions compare favorably with the experimental data in both cases at $\alpha = 0.0735$ and $Q_0 = 0$.

CONCLUSIONS

(1) The temperature distribution and total heat transfer rate are obtained for a fin with and without heat transfer at its end.

(2) For $m \geqslant 1$, the temperature along a fin decreases asymptotically to the ambient temperature in a very long fin.

(3) For $1 > m > 0$, the tip temperature may drop to zero, and no heat transfer at the end portion when N exceeds N_0.

(4) For $m \leqslant 0$, there exists an N_{max}, which is related to the thermal instability. No solution is found provided that N is greater than N_{max}.

(5) For $m \leqslant 0$, the stability condition becomes stricter for a smaller m and a larger Q_0 or α.

(6) The predicted temperature profiles and total heat transfer rate compare well with experimental data in film and nucleate boiling with water and isopropyl alcohol.

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